The tensor bi-spectrum in a matter bounce

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Abstract. Matter bounces are bouncing scenarios wherein the universe contracts as in a matter dominated phase at early times. Such scenarios are known to lead to a scale invariant spectrum of tensor perturbations, just as de Sitter inflation does. In this work, we examine if the tensor bi-spectrum can discriminate between the inflationary and the bouncing scenarios. Using the Maldacena formalism, we analytically evaluate the tensor bi-spectrum in a matter bounce for an arbitrary triangular configuration of the wavevectors. We show that, over scales of cosmological interest, the non-Gaussianity parameter $h_{\rm NL}$ that characterizes the amplitude of the tensor bi-spectrum is quite small when compared to the corresponding values in de Sitter inflation. During inflation, the amplitude of the tensor perturbations freeze on super-Hubble scales, a behavior that results in the so-called consistency condition relating the tensor bi-spectrum and the power spectrum in the squeezed limit. In contrast, in the bouncing scenarios, the amplitude of the tensor perturbations grow strongly as one approaches the bounce, which suggests that the consistency condition will not be valid in such situations. We explicitly show that the consistency relation is indeed violated in the matter bounce. We discuss the implications of the results.

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Contents

1 Introduction

Bouncing models correspond to situations wherein the universe initially goes through a period of contraction until the scale factor reaches a certain minimum value before transiting to the expanding phase. They offer an alternative to inflation to overcome the horizon problem, as they permit well motivated, Minkowski-like initial conditions to be imposed on the perturbations at early times during the contracting phase (see, for instance, Refs. [1–18]; for reviews, see Refs. [19–21]). Interestingly, certain bouncing scenarios can lead to nearly scale invariant perturbation spectra (see, for example, Refs. [3, 11, 18]), as is required by observations [22, 23]. For instance, a bouncing model wherein the universe goes through a contracting phase as driven by matter—referred to as a matter bounce—is known to lead to an exactly scale invariant spectrum of tensor perturbations as in de Sitter inflation [1–3]. Clearly, it will be worthwhile to examine if non-Gaussianities can help us discriminate between such scenarios [24–26].

The most dominant of the non-Gaussian signatures are the non-vanishing three-point functions involving the scalars as well as the tensors [27–34]. In order to drive a bounce, it is well known that one requires matter fields that violate the null energy condition. Therefore, analyzing the evolution of the scalar perturbations require suitable modelling of the matter fields [19–21]. In contrast, the tensor perturbations depend only on the scale factor and hence are simpler to study. For this reason, we shall focus on the tensor bi-spectrum in this work. Further, we shall assume a specific functional form for the scale factor and we shall not attempt to construct sources that can give rise to such a behavior.

An interesting aspect of the three-point functions is their property in the so-called squeezed limit wherein the wavelength of one of the three modes involved is much larger than the other two [27, 35–43]. In this limit, under fairly general conditions, it is known that the three-point functions can be expressed completely in terms of the two-point functions, a relation that is referred to as the consistency condition. We should mention that, while the scalar consistency relation has drawn most of the attention, it has been established that all the four three-point functions involving scalars and tensors satisfy similar relations under certain conditions [28–30, 44, 45]. It is interesting to examine if the three-point functions generated in the bouncing scenarios satisfy the consistency condition. In the context of inflation, it is well known that the consistency relations arise due to the fact that the amplitude

of the long wavelength mode freezes on super-Hubble scales. In contrast, in a bouncing universe it can be readily shown that the amplitude of the long wavelength mode grows sharply as one approaches the bounce during the contracting phase. This behavior suggests that the consistency relation may not hold in bouncing models [24]. The primordial consistency conditions lead to corresponding imprints on the anisotropies in the cosmic microwave background (see, for instance, Refs. [46–48]; in particular, see Ref. [49] for the signatures of the tensor modes) and the large scale structure (see, for example, Refs. [50–52]). It is clear that the consistency condition, if it can be confirmed by the observations, can help us discriminate between models of the early universe.

The most comprehensive formalism to study the generation of non-Gaussianities in the early universe is the approach due to Maldacena [27]. In this work, we analytically evaluate the tensor bi-spectrum in a matter bounce using the Maldacena formalism. To arrive at the tensor bi-spectrum analytically, one requires not only the behavior of the tensor modes, one also needs to be able to evaluate a certain integral involving the scale factor and the tensor modes. We conveniently divide the evolution into three domains and use the analytic solutions available in these domains to carry out the integrals and obtain the tensor bi-spectrum.

This paper is organized as follows. In the following section, considering a specific form for the scale factor, we shall divide the period before the bounce into two domains, the first corresponding to early times during the contracting phase and the other close to the bounce. We shall describe the analytic solutions to the tensor modes during these two domains and also discuss the behavior of the modes after the bounce to arrive at the corresponding tensor power spectrum over wavenumbers much smaller than the wavenumber associated with the bounce. We shall also compare the analytical solutions for the tensor modes with the corresponding results obtained numerically. In Sec. 3, we shall quickly summarize the essential expressions describing the tensor bi-spectrum in the Maldacena formalism. We shall also introduce the tensor non-Gaussianity parameter $h_{\rm NL}$, a dimensionless quantity that reflects the amplitude of the tensor bi-spectrum. In Sec. 4, we shall evaluate the tensor bi-spectrum using the analytic solutions to the modes and the behavior of the scale factor in the three domains. We shall calculate the bi-spectrum for an arbitrary triangular configuration of the wavevectors. In Sec. 5, we shall illustrate the results in the equilateral and the squeezed limits. We shall show that the non-Gaussianity parameter $h_{\rm NL}$ that characterizes the tensor bi-spectrum is very small for cosmological scales and that the consistency relation is violated in the squeezed limit. We shall conclude with a brief discussion in Sec. 6. In an Appendix, we shall briefly outline a proof of the consistency condition satisfied by the tensor bi-spectrum during inflation.

Note that we shall work with natural units wherein $\hbar = c = 1$, and define the Planck mass to be $M_{\rm Pl} = (8\,\pi\,G)^{-1/2}$.

2 The tensor modes and the power spectrum

We shall consider the background to be the spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric that is described by the line-element

$$ {\rm d}s^2 = a^2(\eta)\,\left(-{\rm d}\eta^2 + \delta_{ij}\,{\rm d}\boldsymbol{x}^i\,{\rm d}\boldsymbol{x}^j\right), \tag{2.1} $$

where $a(\eta)$ denotes the scale factor and η is the conformal time coordinate. We shall assume that the scale factor describing the bouncing scenario is given in terms of the conformal time

coordinate η by the relation

$$a(\eta) = a_0\,\left(1+\eta^2/\eta_0^2\right) = a_0\,\left(1+k_0^2\,\eta^2\right), \tag{2.2}$$

where a_0 is the minimum value of the scale factor at the bounce (*i.e.* when $\eta=0$) and $\eta_0 = k_0^{-1}$ denotes the time scale that determines the duration of the bounce. Note that, at very early times, *viz.* when $\eta \ll -\eta_0$, the scale factor behaves as in a matter dominated universe (*i.e.* as $a \propto \eta^2$) and, for this reason, such a bouncing model is often referred to as the matter bounce. We shall assume that the scale associated with the bounce, *viz.* k_0, is of the order of the Planck scale $M_{_{\mathrm{Pl}}}$. Therefore, the wavenumbers of cosmological interest are 50–60 orders of magnitude smaller than the wavenumber k_0.

Upon taking into account the tensor perturbations characterized by γ_{ij}, the spatially flat FLRW metric can be expressed as [27]

$$\mathrm{d}s^2 = a^2(\eta)\,\left(-\mathrm{d}\eta^2 + \left[\mathrm{e}^{\gamma(\eta,\boldsymbol{x})}\right]_{ij}\mathrm{d}\boldsymbol{x}^i\,\mathrm{d}\boldsymbol{x}^j\right). \tag{2.3}$$

Recall that the primordial perturbations are generated due to quantum fluctuations. On quantization, the tensor perturbation $\hat{\gamma}_{ij}$ can be decomposed in terms of the corresponding Fourier modes h_k as follows:

$$\begin{aligned}\hat{\gamma}_{ij}(\eta,\mathbf{x}) &= \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\hat{\gamma}_{ij}^{\boldsymbol{k}}(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}}\\ &= \sum_s \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\,\pi)^{3/2}}\,\left(\hat{a}_{\boldsymbol{k}}^s\,\varepsilon_{ij}^s(\boldsymbol{k})\,h_k(\eta)\,\mathrm{e}^{i\,\boldsymbol{k}\cdot\boldsymbol{x}} + \hat{a}_{\mathbf{k}}^{s\dagger}\,\varepsilon_{ij}^{s*}(\boldsymbol{k})\,h_k^*(\eta)\,\mathrm{e}^{-i\,\boldsymbol{k}\cdot\boldsymbol{x}}\right).\end{aligned} \tag{2.4}$$

In this decomposition, the pair of operators $(\hat{a}_{\boldsymbol{k}}^s, \hat{a}_{\boldsymbol{k}}^{s\dagger})$ represent the annihilation and creation operators corresponding to the tensor modes associated with the wavevector $\boldsymbol{k}$, and they satisfy the standard commutation relations. The quantity $\varepsilon_{ij}^s(\boldsymbol{k})$ represents the polarization tensor of the gravitational waves with their helicity being denoted by the index s. The transverse and traceless nature of the gravitational waves leads to the conditions $\varepsilon_{ii}^s(\boldsymbol{k}) = k_i\,\varepsilon_{ij}^s(\boldsymbol{k}) = 0$. We shall work with a normalization such that $\varepsilon_{ij}^r(\boldsymbol{k})\,\varepsilon_{ij}^{s*}(\boldsymbol{k}) = 2\,\delta^{rs}$ [27]. The tensor power spectrum, *viz.* $\mathcal{P}_{_{\mathrm{T}}}(k)$, is defined as follows:

$$\langle\,\hat{\gamma}_{ij}^{\boldsymbol{k}}(\eta_{\mathrm{e}})\,\hat{\gamma}_{mn}^{\boldsymbol{k}'}(\eta_{\mathrm{e}})\,\rangle = \frac{(2\,\pi)^2}{2\,k^3}\,\frac{\Pi_{ij,mn}^{\boldsymbol{k}}}{4}\,\mathcal{P}_{_{\mathrm{T}}}(k)\,\delta^{(3)}(\boldsymbol{k}+\boldsymbol{k}'), \tag{2.5}$$

where the expectation values on the left hand sides are to be evaluated in the specified initial quantum state of the perturbations, and η_{e} denotes a suitably late conformal time when the power spectrum is to be evaluated. The quantity $\Pi_{ij,mn}^{\boldsymbol{k}}$ is given by [28–30, 44, 45]

$$\Pi_{ij,mn}^{\boldsymbol{k}} = \sum_s\,\varepsilon_{ij}^s(\boldsymbol{k})\,\varepsilon_{mn}^{s*}(\boldsymbol{k}). \tag{2.6}$$

The tensor spectral index $n_{_{\mathrm{T}}}$ is defined as

$$n_{_{\mathrm{T}}} = \frac{\mathrm{d}\ln \mathcal{P}_{_{\mathrm{T}}}(k)}{\mathrm{d}\ln k}. \tag{2.7}$$

The tensor modes h_k satisfy the differential equation

$$h_k'' + 2\,\frac{a'}{a}\,h_k' + k^2\,h_k = 0, \tag{2.8}$$

where the overprimes denote differentiation with respect to the conformal time η. If we write $h_k = u_k/a$, then the modes u_k satisfy the equation

$$u_k'' + \left(k^2 - \frac{a''}{a}\right) u_k = 0. \tag{2.9}$$

The quantity a''/a corresponding to the scale factor (2.2) is given by

$$\frac{a''}{a} = \frac{2\,k_0^2}{1 + k_0^2\,\eta^2}, \tag{2.10}$$

which has essentially a Lorentzian profile. Note that the quantity a''/a exhibits a maximum at the bounce, with the maximum value being of the order of k_0^2. Also, it goes to zero as $\eta \to \pm\infty$. For modes of cosmological interest, one finds that $k^2 \gg a''/a$ at suitably early times (*i.e.* as $\eta \to -\infty$). In this domain, the quantity u_k oscillates and we can impose the standard initial conditions on these modes and study their evolution thereafter.

Let us divide the period before the bounce into two domains, a domain corresponding to early times and another closer to the bounce. Let the first domain be determined by the condition $-\infty < \eta < -\alpha\,\eta_0$, where α is a relatively large number, which we shall set to be, say, 10^5. The second domain evidently corresponds to $-\alpha\,\eta_0 < \eta < 0$. In the first domain, we can assume that the scale factor behaves as

$$a(\eta) \simeq a_0\,k_0^2\,\eta^2, \tag{2.11}$$

so that $a''/a = 2/\eta^2$. Since the condition $k^2 = a''/a$ corresponds to, say, $\eta_k = -\sqrt{2}/k$, the initial conditions can be imposed when $\eta \ll \eta_k$. The modes h_k can be easily obtained in such a case and the positive frequency modes that correspond to the vacuum state at early times are given by [1–3]

$$h_k = \frac{\sqrt{2}}{M_{_{\rm Pl}}}\,\frac{1}{\sqrt{2\,k}}\,\frac{1}{a_0\,k_0^2\,\eta^2}\,\left(1 - \frac{i}{k\,\eta}\right)\,{\rm e}^{-i\,k\,\eta}. \tag{2.12}$$

Let us now consider the behavior of the modes in the second domain, *i.e.* when $-\alpha\,\eta_0 < \eta < 0$. Since we are interested in scales much smaller than k_0, we shall assume that $\eta_k \ll -\alpha\,\eta_0$, which corresponds to the condition $k \ll k_0/\alpha$. Therefore, in this domain, for scales of cosmological interest, the equation governing the tensor mode h_k reduces to

$$h_k'' + \frac{2\,a'}{a}\,h_k' \simeq 0. \tag{2.13}$$

This equation can be immediately integrated to yield

$$h_k'(\eta) \simeq h_k'(\eta_\ast)\,\frac{a^2(\eta_\ast)}{a^2(\eta)}, \tag{2.14}$$

where $\eta_\ast$ is a suitably chosen time and the scale factor $a(\eta)$ is given by the complete expression (2.2). On further integration, we obtain that

$$h_k(\eta) = h_k(\eta_\ast) + h_k'(\eta_\ast)\,a^2(\eta_\ast)\,\int_{\eta_\ast}^{\eta} \frac{{\rm d}\tilde{\eta}}{a^2(\tilde{\eta})}, \tag{2.15}$$

where we have chosen the constant of integration to be $h_k(\eta_\ast)$. If we choose $\eta_\ast = -\alpha\,\eta_0$, we can make use of the solution (2.12) to determine $h_k(\eta_\ast)$ and $h_k'(\eta_\ast)$. Note that, in the

domain of interest, the first term in the above expression is, evidently, a constant, while the second term grows rapidly as one approaches the bounce. Upon using the form (2.2) of the scale factor, we find that we can express the behavior of the mode h_k in the second domain as

$$h_k = A_k + B_k\, f(k_0\, \eta), \tag{2.16}$$

where

$$f(k_0\, \eta) = \frac{k_0\, \eta}{1 + k_0^2\, \eta^2} + \tan^{-1}(k_0\, \eta)\,, \tag{2.17}$$

while the quantities A_k and B_k are given by

$$A_k = \frac{\sqrt{2}}{M_{\rm Pl}}\, \frac{1}{\sqrt{2\, k}}\, \frac{1}{a_0\, \alpha^2}\, \left(1 + \frac{i\, k_0}{\alpha\, k}\right)\, {\rm e}^{i\, \alpha\, k/k_0} + B_k\, f(\alpha), \tag{2.18}$$

$$B_k = \frac{\sqrt{2}}{M_{\rm Pl}}\, \frac{1}{\sqrt{2\, k}}\, \frac{1}{2\, a_0\, \alpha^2}\, \left(1 + \alpha^2\right)^2\, \left(\frac{3\, i\, k_0}{\alpha^2\, k} + \frac{3}{\alpha} - \frac{i\, k}{k_0}\right)\, {\rm e}^{i\, \alpha\, k/k_0}. \tag{2.19}$$

Let us now turn to the third domain, *i.e.* immediately after the bounce. In this case too, for modes such that $k \ll k_0/\alpha$, the solution to h_k is given by Eq. (2.16). We should highlight the fact that, whereas the bounce (2.2) is a symmetric one, the solution (2.16) is asymmetric in η. Moreover, one may have naively expected the amplitude of the long wavelength modes to freeze once the universe starts expanding. This is largely true, though not completely so, and the behavior can possibly be attributed to the specific form of the scale factor (2.2). Note that, during this domain, while the first term in $f(k_0\, \eta)$ decays, the second term actually grows, albeit extremely mildly. We shall assume that, after the bounce, the universe transits to the conventional radiation domination epoch at, say, $\eta = \beta\, \eta_0$, where we shall set $\beta \simeq 10^2$. We should mention that this choice is somewhat arbitrary and we shall discuss the dependence of the tensor power spectrum and the bi-spectrum on β in due course.

In order to understand the extent of accuracy of the approximations involved, it would be worthwhile to compare the above analytical results for the mode h_k with the corresponding numerical results. Clearly, given the scale factor (2.2), it is a matter of integrating the differential equation (2.8), along with the standard Bunch-Davies initial conditions, to arrive at the behavior of h_k. The conformal time coordinate does not prove to be an efficient time variable for numerical integration, particularly when a large range in the scale factor needs to be covered. In the context of inflation, it is often the e-fold N, defined as $a(N) = a_0 \exp N$, where $N = 0$ is a suitable time at which the scale factor takes the value a_0, that is utilized to integrate the equations of motion (see, for instance, Refs. [53–56]). Due to the exponential factor involved, a small range in e-folds covers a large range in time and scale factor. However, since ${\rm e}^N$ is a monotonically increasing function, while it is useful to describe expanding universes, e-folds are not helpful in characterizing bouncing scenarios. In order to characterize a bounce, it would be convenient to choose a variable that is negative during the contracting phase of the universe, zero at the bounce and positive during the expanding phase. We shall choose to perform the integration using a new variable $\mathcal{N}$, which we call the e-N-fold, in terms of which the scale factor is defined as $a(\mathcal{N}) = a_0 \exp\left(\mathcal{N}^2/2\right)$ [57]. We shall assume that $\mathcal{N}$ is zero at the bounce, with negative values representing the phase prior to the bounce and positive values after.

In terms of the e-N-fold, the differential equation (2.8) governing the evolution of the tensor modes can be expressed as

$$\frac{\mathrm{d}^2 h_k}{\mathrm{d}\mathcal{N}^2} + \left(3\,\mathcal{N} + \frac{1}{H}\,\frac{\mathrm{d}H}{\mathrm{d}\mathcal{N}} - \frac{1}{\mathcal{N}}\right)\frac{\mathrm{d}h_k}{\mathrm{d}\mathcal{N}} + \left(\frac{k\,\mathcal{N}}{a\,H}\right)^2 h_k = 0, \tag{2.20}$$

where $H = a'/a^2$ is the Hubble parameter. Given the scale factor (2.2), the corresponding Hubble parameter can be easily evaluated in terms of the conformal time η. In order to express the Hubble parameter H in terms of the e-N-fold, we shall require η as a function of $\mathcal{N}$. Upon using the definition of the e-N-folds and the expression (2.2) for the scale factor, we obtain that

$$\eta(\mathcal{N}) = \pm\, k_0^{-1}\, \left(\mathrm{e}^{\mathcal{N}^2/2} - 1\right)^{1/2}. \tag{2.21}$$

Since the Hubble parameter is negative during the contracting phase and positive during the expanding regime, we have to choose the root of $\eta(\mathcal{N})$ accordingly during each phase. From the expression for the Hubble parameter, we evaluate the coefficients of the differential equation (2.20) in terms of $\mathcal{N}$. With the coefficients at hand, we numerically integrate the differential equation using a fifth order Runge-Kutta algorithm. We should mention that we have also independently checked the numerical results using *Mathematica*. Recall that, the initial conditions are imposed in a domain during the contracting phase wherein $k^2 \gg a''/a$. As is done in the context of inflation, we shall impose the initial conditions when $k^2 = 10^4\,(a''/a)$ corresponding to, say, the e-N-fold $\mathcal{N}_i$. It should be pointed out that the initial conditions on the different modes are imposed at different times. In terms of the e-N-folds, the standard Bunch-Davies initial conditions can be expressed as

$$h_k = \frac{1}{a(\mathcal{N}_i)\sqrt{2k}}, \tag{2.22a}$$

$$\frac{\mathrm{d}h_k}{\mathrm{d}\mathcal{N}} = -\frac{i\,\mathcal{N}_i}{a^2(\mathcal{N}_i)\,H(\mathcal{N}_i)}\,\sqrt{\frac{k}{2}} - \frac{\mathcal{N}_i}{a(\mathcal{N}_i)\,\sqrt{2k}}. \tag{2.22b}$$

We impose these initial conditions well before the bounce and evolve the modes until a suitable time after the bounce. The tensor mode h_k evaluated in such a manner has been plotted for a given wavenumber (such that $k/k_0 \ll 1$) in Fig. 1. The figure also contains a plot of the analytical results (2.12) and (2.16) for the same wavenumber. As is evident from the figure, prior to the bounce and immediately after, the analytical results match the exact numerical results exceedingly well.

The tensor power spectrum after the bounce can be calculated using the solutions we have obtained. Recall that the tensor power spectrum is defined as

$$\mathcal{P}_{\mathrm{T}}(k) = 4\,\frac{k^3}{2\,\pi^2}\,|h_k(\eta)|^2, \tag{2.23}$$

with the spectrum to be evaluated at a suitable time. If we evaluate the tensor power spectrum at $\eta = \beta\,\eta_0$, we find that it can be expressed as

$$\mathcal{P}_{\mathrm{T}}(k) = 4\,\frac{k^3}{2\,\pi^2}\,|A_k + B_k\,f(\beta)|^2. \tag{2.24}$$

Note that, α is a quantity that we have artificially introduced and the actual problem does not contain α. For $k \ll k_0/\alpha$ and a sufficiently large α (as we had said, for $\alpha = 10^5$ or so),

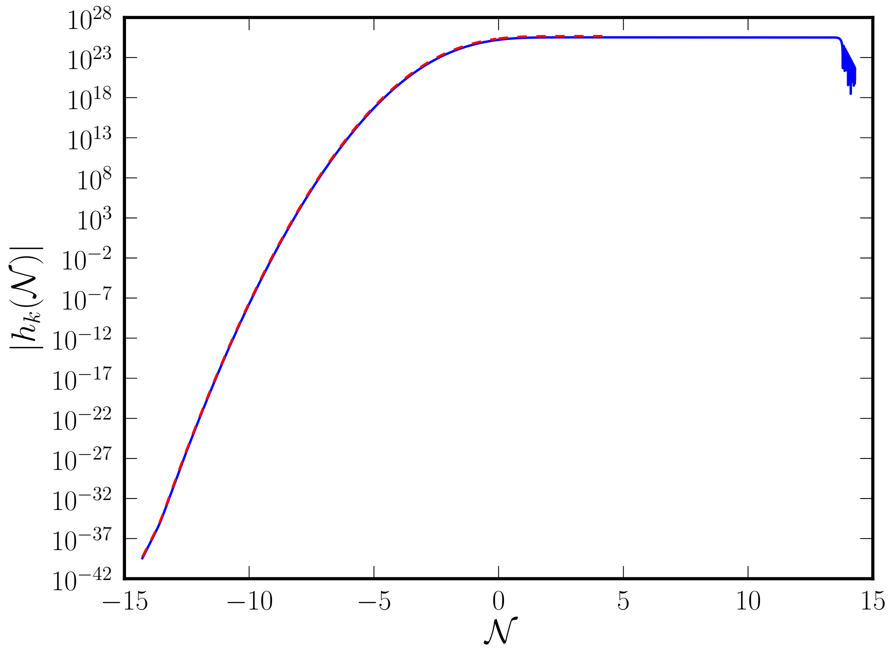

Figure 1. A comparison of the numerical results (in blue) with the analytical results (in red) for the amplitude of the tensor mode $|h_k|$ corresponding to the wavenumber $k/k_0 = 10^{-20}$. We have set $a_0 = 10^5$ and, for plotting the analytical results, we have chosen $\alpha = 10^5$. We have plotted the results from the initial e-N-fold $\mathcal{N}_i$ [when $k^2 = 10^4\,(a''/a)$] corresponding to the mode. While we have illustrated the exact numerical result till rather late times, we have plotted the analytical results until a time after the bounce when the power spectrum is evaluated (see discussion below). Evidently, the analytical and numerical results match extremely well, suggesting that the analytical approximation for the modes works to a very good accuracy.

the above power spectrum reduces to a scale invariant form with a weak dependence on β, if β is reasonably larger than unity. If we further assume that β is large enough (say, 10^2), then the scale invariant amplitude is found to be: $\mathcal{P}_{\rm T}(k) \simeq 9\,k_0^2/(2\,M_{\rm Pl}^2\,a_0^2)$, as expected [1–3]. In Fig. 2, we have plotted the complete tensor power spectrum described by the expression (2.24) for a given set of parameters. We should stress that the power spectrum is actually valid only for modes which satisfy the condition $k \ll k_0/\alpha$. It is evident from the figure that the power spectrum is strictly scale invariant over this domain. Moreover, we find that the spectrum indeed reduces to the above-mentioned scale invariant amplitude for small values of the wavenumbers. We have also evaluated the tensor power spectrum numerically using the method described above. We have computed the spectrum at a given time soon after the bounce (corresponding to $\beta = 10^2$) for all the modes. We find that, for wavenumbers such that $k \ll k_0$, the numerical analysis also leads to a scale invariant spectrum whose amplitude matches the above analytical result to about 1%.

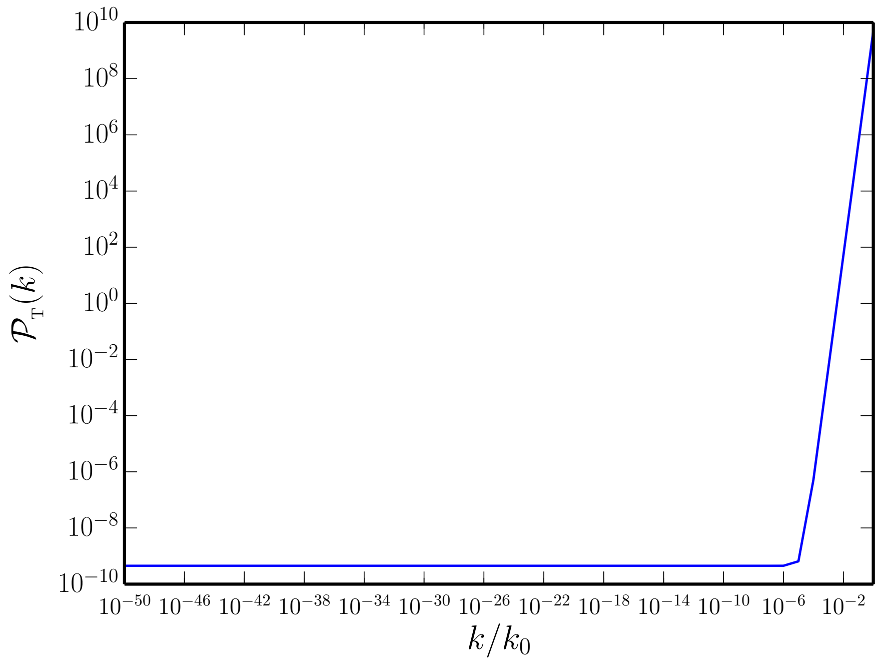

Figure 2. The behavior of the tensor power spectrum has been plotted as a function of k/k_0 for a wide range of wavenumbers. In plotting this figure, we have set $k_0/M_{_{\rm Pl}} = 1$, $a_0 = 10^5$, $\alpha = 10^5$ and $\beta = 10^2$. We should emphasize that the approximations we have worked with are valid only over the domain wherein $k \ll k_0/\alpha$. Clearly, the power spectrum is scale invariant in this domain. We also find that, at small wavenumbers, the tensor power spectrum has the expected scale invariant amplitude of $\mathcal{P}_{_{\rm T}}(k) = 4.5 \times 10^{-10}$ corresponding to $k_0/M_{_{\rm Pl}} = 1$ and $a_0 = 10^5$.

3 The tensor bi-spectrum and the corresponding non-Gaussianity parameter

As we have mentioned, the most comprehensive formalism to calculate the three-point functions generated in the early universe is the formalism due to Maldacena [27]. The primary aim of Maldacena's approach is to obtain the cubic order action that governs the scalar and the tensor perturbations using the ADM formalism. Then, based on the action, one arrives at the corresponding three-point functions using the standard rules of perturbative quantum field theory.

The tensor bi-spectrum in Fourier space, *viz.* $\mathcal{B}^{m_1n_1m_2n_2m_3n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, evaluated at the conformal time, say, $\eta_{\rm e}$, is defined as

$$\langle\, \hat{\gamma}^{\boldsymbol{k}_1}_{m_1n_1}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_2}_{m_2n_2}(\eta_{\rm e})\, \hat{\gamma}^{\boldsymbol{k}_3}_{m_3n_3}(\eta_{\rm e})\, \rangle \equiv (2\,\pi)^3\; \mathcal{B}^{m_1n_1m_2n_2m_3n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\; \delta^{(3)}\, (\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\,. \tag{3.1}$$

Note that the delta function on the right hand side implies that the wavevectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ form the edges of a triangle. For convenience, hereafter, we shall set

$$\mathcal{B}^{m_1n_1m_2n_2m_3n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = (2\,\pi)^{-9/2}\; G^{m_1n_1m_2n_2m_3n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3). \tag{3.2}$$

The tensor bi-spectrum $G^{m_1n_1m_2n_2m_3n_3}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$, calculated in the perturbative vacuum using the Maldacena formalism, can be written in terms of the modes h_k as follows [27, 31–

34, 44]:

$$G_{\gamma\gamma\gamma}^{m_1n_1m_2n_2m_3n_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = M_{_{\rm Pl}}^2 \biggl[\Bigl(\Pi^{\boldsymbol{k}_1}_{m_1n_1,ij}\,\Pi^{\boldsymbol{k}_2}_{m_2n_2,im}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,lj} - \frac{1}{2}\,\Pi^{\boldsymbol{k}_1}_{m_1n_1,ij}\,\Pi^{\boldsymbol{k}_2}_{m_2n_2,ml}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,ij}\Bigr)\,k_{1m}\,k_{1l} + \text{five permutations}\biggr] \times \bigl[h_{k_1}(\eta_{\rm e})\,h_{k_2}(\eta_{\rm e})\,h_{k_3}(\eta_{\rm e})\,\mathcal{G}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) + \text{complex conjugate}\bigr], \tag{3.3}$$

where $\mathcal{G}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ is described by the integral

$$\mathcal{G}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\frac{i}{4}\int_{\eta_{\rm i}}^{\eta_{\rm e}} {\rm d}\eta\; a^2\, h_{k_1}^\ast\, h_{k_2}^\ast\, h_{k_3}^\ast, \tag{3.4}$$

with $\eta_{\rm i}$ denoting the time when the initial conditions are imposed on the perturbations and $\eta_{\rm e}$ representing the time when the bi-spectrum is to be evaluated. Also, we should mention that (k_{1i}, k_{2i}, k_{3i}) denote the components of the three wavevectors $(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ along the i-spatial direction[1].

The dimensionless non-Gaussianity parameter that characterizes the amplitude of the tensor bi-spectrum is defined as [34]

$$h_{_{\rm NL}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\left(\frac{4}{2\,\pi^2}\right)^2 \left[k_1^3\,k_2^3\,k_3^3\,G_{\gamma\gamma\gamma}^{m_1n_1m_2n_2m_3n_3}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\right] \times \left[\Pi^{\boldsymbol{k}_1}_{m_1n_1,m_3n_3}\,\Pi^{\boldsymbol{k}_2}_{m_2n_2,\bar{m}\bar{n}}\,k_3^3\,\mathcal{P}_{_{\rm T}}(k_1)\,\mathcal{P}_{_{\rm T}}(k_2) + \text{five permutations}\right]^{-1}, \tag{3.5}$$

where the overbars on the indices imply that they need to be summed over all allowed values. Our aim in this work is to evaluate the magnitude and shape of the tensor bi-spectrum and the corresponding non-Gaussianity parameter and compare with, say, the results in de Sitter inflation. Therefore, for simplicity, we shall set the polarization tensor to unity. In such a case, the expression (3.3) for the tensor bi-spectrum above simplifies to

$$G_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = M_{_{\rm Pl}}^2\,\bigl[h_{k_1}(\eta_{\rm e})\,h_{k_2}(\eta_{\rm e})\,h_{k_3}(\eta_{\rm e})\,\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) + \text{complex conjugate}\bigr], \tag{3.6}$$

where the quantity $\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ is described by the integral

$$\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\frac{i}{4}\,\left(k_1^2+k_2^2+k_3^2\right)\int_{\eta_{\rm i}}^{\eta_{\rm e}} {\rm d}\eta\; a^2\, h_{k_1}^\ast\, h_{k_2}^\ast\, h_{k_3}^\ast. \tag{3.7}$$

We shall choose $\eta_{\rm i}$ to be an early time during the contracting phase when the initial conditions are imposed on the modes (*i.e.* when $k^2 \gg a''/a$), and $\eta_{\rm e}$ to be a suitably late time, say, some time after the bounce, when the bi-spectrum is evaluated. If we ignore the factors involving the polarization tensor, the non-Gaussianity parameter $h_{_{\rm NL}}$ reduces to

$$h_{_{\rm NL}}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = -\left(\frac{4}{2\,\pi^2}\right)^2 \left[k_1^3\,k_2^3\,k_3^3\,G_{\gamma\gamma\gamma}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\right] \times \left[2\,k_3^3\,\mathcal{P}_{_{\rm T}}(k_1)\,\mathcal{P}_{_{\rm T}}(k_2) + \text{two permutations}\right]^{-1}. \tag{3.8}$$

[1]Such a clarification seems necessary to avoid confusion between k_1, k_2 and k_3 which denote the wavenumbers associated with the wavevectors $\boldsymbol{k}_1$, $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$, and the quantity k_i which represents the component of the wavevector $\boldsymbol{k}$ along the i-spatial direction.

4 Evaluating the tensor bi-spectrum

With the forms of the scale factor and the mode functions at hand, in order to arrive at the tensor bi-spectrum, it is now a matter of evaluating the integral (3.7) in the three domains.

Let us begin by considering the first domain. Upon using the behavior (2.11) of the scale factor and the mode (2.12) in the first domain, we find that the quantity $\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ can be expressed as

$$\begin{aligned}\bar{\mathcal{G}}^1_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = \frac{-i\,\left(k_1^2 + k_2^2 + k_3^2\right)}{4\,M_{_{\mathrm{Pl}}}^3\, a_0\, k_0^2\, \sqrt{k_1\,k_2\,k_3}} \biggl[I_2(k_{_{\mathrm{T}}}, k_0, \alpha) + i\,\left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right)\, I_3(k_{_{\mathrm{T}}}, k_0, \alpha) \\ - \left(\frac{1}{k_1\,k_2} + \frac{1}{k_2\,k_3} + \frac{1}{k_1\,k_3}\right)\, I_4(k_{_{\mathrm{T}}}, k_0, \alpha) - \frac{i}{k_1\,k_2\,k_3}\, I_5(k_{_{\mathrm{T}}}, k_0, \alpha)\biggr], \end{aligned} \tag{4.1}$$

where $k_{_{\mathrm{T}}} = k_1 + k_2 + k_3$ and the quantities $I_n(k_{_{\mathrm{T}}}, k_0, \alpha)$ are described by the integrals

$$I_n(k_{_{\mathrm{T}}}, k_0, \alpha) = \int_{-\infty}^{-\alpha/k_0} \frac{\mathrm{d}\eta}{\eta^n}\, \mathrm{e}^{i\,k_{_{\mathrm{T}}}\,\eta}. \tag{4.2}$$

For $n > 1$, these integrals can be evaluated to yield

$$I_{n+1}(k_{_{\mathrm{T}}}, k_0, \alpha) = -\frac{1}{n}\left(-\frac{k_0}{\alpha}\right)^n \mathrm{e}^{-i\,\alpha\,k_{_{\mathrm{T}}}/k_0} + \frac{i\,k_{_{\mathrm{T}}}}{n}\, I_n(k_{_{\mathrm{T}}}, k_0, \alpha), \tag{4.3}$$

while $I_1(k_{_{\mathrm{T}}}, k_0, \alpha)$ is given by

$$I_1(k_{_{\mathrm{T}}}, k_0, \alpha) = i\,\pi + \mathrm{Ei}(-i\,\alpha\,k_{_{\mathrm{T}}}/k_0), \tag{4.4}$$

where $\mathrm{Ei}(x)$ is the exponential integral function [58].

Let us now turn to evaluating $\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ in the second domain. Upon using the behavior (2.2) of the scale factor and the mode (2.16), we find that the quantity can be expressed as

$$\begin{aligned}\bar{\mathcal{G}}^2_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) = -\frac{i\,a_0^2\,\left(k_1^2 + k_2^2 + k_3^2\right)}{4\,k_0} \biggl[&A_{k_1}^\ast\, A_{k_2}^\ast\, A_{k_3}^\ast\, J_0(\alpha) \\ &+ \left(A_{k_1}^\ast\, A_{k_2}^\ast\, B_{k_3}^\ast + A_{k_1}^\ast\, B_{k_2}^\ast\, A_{k_3}^\ast + B_{k_1}^\ast\, A_{k_2}^\ast\, A_{k_3}^\ast\right)\, J_1(\alpha) \\ &+ \left(A_{k_1}^\ast\, B_{k_2}^\ast\, B_{k_3}^\ast + B_{k_1}^\ast\, A_{k_2}^\ast\, B_{k_3}^\ast + B_{k_1}^\ast\, B_{k_2}^\ast\, A_{k_3}^\ast\right)\, J_2(\alpha) \\ &+ B_{k_1}^\ast\, B_{k_2}^\ast\, B_{k_3}^\ast\, J_3(\alpha)\biggr], \end{aligned} \tag{4.5}$$

where $J_n(\alpha)$ are described by the integrals

$$J_n(\alpha) = \int_{-\alpha}^{0} \mathrm{d}x\, \left(1 + x^2\right)^2\, f^n(x), \tag{4.6}$$

with the function $f(x)$ being given by Eq. (2.17). The integrals $J_0(\alpha)$ and $J_1(\alpha)$ can be readily evaluated to obtain that

$$J_0(\alpha) = \alpha + \frac{2\,\alpha^3}{3} + \frac{\alpha^5}{5} \tag{4.7}$$

and

$$
\begin{aligned}
J_1(\alpha) &= -\frac{1}{2}\left(\alpha^2+\frac{\alpha^4}{2}\right)-\frac{11}{60}+\frac{2\,(1+\alpha^2)}{15}+\frac{\left(1+\alpha^2\right)^2}{20}-\frac{8\,\alpha}{15}\,\tan^{-1}\alpha \\
&\quad -\frac{4\,\alpha}{15}\,\left(1+\alpha^2\right)\,\tan^{-1}\alpha-\frac{\alpha}{5}\,\left(1+\alpha^2\right)^2\,\tan^{-1}\alpha+\frac{4}{15}\ln\left(1+\alpha^2\right). \qquad (4.8)
\end{aligned}
$$

In contrast, the integrals $J_2(\alpha)$ and $J_3(\alpha)$ are more involved. The integral $J_2(\alpha)$ can be divided into three parts and written as

$$
J_2(\alpha) = J_{21}(\alpha) + J_{22}(\alpha) + J_{23}(\alpha), \qquad (4.9)
$$

where the integrals $J_{21}(\alpha)$ and $J_{22}(\alpha)$ can be easily evaluated to be

$$
J_{21}(\alpha) = \int_{-\alpha}^{0} \mathrm{d}x\, x^2 = \frac{\alpha^3}{3}, \qquad (4.10)
$$

$$
\begin{aligned}
J_{22}(\alpha) &= 2\int_{-\alpha}^{0}\mathrm{d}x\, x\,\left(1+x^2\right)\,\tan^{-1}x \\
&= \alpha^2\left(1+\frac{\alpha^2}{2}\right)\,\tan^{-1}\alpha-\frac{1}{2}\,\left(\alpha-\tan^{-1}\alpha\right)-\frac{\alpha^3}{6}. \qquad (4.11)
\end{aligned}
$$

The quantity $J_{23}(\alpha)$ is given by

$$
J_{23}(\alpha) = \int_{-\alpha}^{0}\mathrm{d}x\,\left(1+x^2\right)^2\,\left(\tan^{-1}x\right)^2, \qquad (4.12)
$$

and, upon setting $\tan^{-1}x = y$, it reduces to

$$
J_{23}(\alpha) = \int_{-\tan^{-1}\alpha}^{0}\mathrm{d}y\, y^2\,\sec^6 y. \qquad (4.13)
$$

The integral involved can be evaluated to be (see, for instance, Ref. [58])

$$
\begin{aligned}
\int \mathrm{d}y\, y^2\,\sec^6\, y &= \frac{-y\,\left(\cos y-2\,y\,\sin y\right)}{10\,\cos^5 y}-\frac{4\,y\,\left(\cos y-y\,\sin y\right)}{15\,\cos^3 y}+\left(\frac{11}{30}+\frac{8\,y^2}{15}\right)\,\tan y \\
&\quad +\frac{\tan^3 y}{30}+\frac{16}{15}\sum_{n=1}^{\infty}\frac{(-1)^n\,2^{2\,n}\,\left(2^{2\,n}-1\right)\,y^{2\,n+1}}{(2\,n+1)\,(2\,n)!}\,B_{2n}, \qquad (4.14)
\end{aligned}
$$

where B_{2n} are the Bernoulli numbers. Needless to add, this result can be used to arrive at $J_{23}(\alpha)$. We should add that the infinite series in the above expression is convergent, and we find that it can be expressed as follows [59]:

$$
\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n\,2^{2\,n}\,\left(2^{2\,n}-1\right)\,y^{2\,n+1}}{(2\,n+1)\,(2\,n)!}\,B_{2n} &= y\left\{\ln\left[\Gamma\left(1+\frac{y}{\pi}\right)\right]+\ln\left[\Gamma\left(1-\frac{y}{\pi}\right)\right]\right. \\
&\quad \left.-\ln\left[\Gamma\left(1-\frac{2y}{\pi}\right)\right]-\ln\left[\Gamma\left(1+\frac{2y}{\pi}\right)\right]\right\} \\
&\quad +\pi\left\{-\zeta'\left(-1,1+\frac{y}{\pi}\right)+\zeta'\left(-1,1-\frac{y}{\pi}\right)\right. \\
&\quad \left.+\frac{1}{2}\,\zeta'\left(-1,1+\frac{2y}{\pi}\right)-\frac{1}{2}\,\zeta'\left(-1,1-\frac{2y}{\pi}\right)\right\}, \qquad (4.15)
\end{aligned}
$$

where $\zeta'(s,a)$ denotes the derivative of the Hurwitz zeta function $\zeta(s,a)$ with respect to the first argument and $\Gamma(n)$ is the Gamma function.

Let us now evaluate the last of the integrals, *viz.* $J_3(\alpha)$. It proves to be convenient to divide the integral into four parts as follows:

$$J_3(\alpha) = J_{31}(\alpha) + J_{32}(\alpha) + J_{33}(\alpha) + J_{34}(\alpha). \tag{4.16}$$

If we set $\tan^{-1} x = y$, we find that the integrals $J_{31}(\alpha)$, $J_{32}(\alpha)$ and $J_{33}(\alpha)$ can be easily evaluated to be

$$\begin{aligned}
J_{31}(\alpha) &= \int_{-\tan^{-1}\alpha}^{0} \mathrm{d}y\, \tan^3 y = -\frac{\alpha^2}{2} + \frac{1}{2}\ln\left(1+\alpha^2\right), \qquad (4.17)\\
J_{32}(\alpha) &= 3\int_{-\tan^{-1}\alpha}^{0} \mathrm{d}y\, y^2\, \tan y\, \sec^4 y\\
&= -\frac{3}{4}\left(1+\alpha^2\right)^2\left(\tan^{-1}\alpha\right)^2 + \frac{\alpha}{2}\left(1+\alpha^2\right)\tan^{-1}\alpha - \frac{\alpha^2}{4}\\
&\quad + \alpha\,\tan^{-1}\alpha - \frac{1}{2}\ln\left(1+\alpha^2\right), \qquad (4.18)\\
J_{33}(\alpha) &= 3\int_{-\tan^{-1}\alpha}^{0} \mathrm{d}y\, y\, \tan^2 y\, \sec^2 y = \frac{\alpha^2}{2} - \alpha^3\,\tan^{-1}\alpha - \frac{1}{2}\ln\left(1+\alpha^2\right). \qquad (4.19)
\end{aligned}$$

The integral $J_{34}(\alpha)$ is given by

$$J_{34}(\alpha) = \int_{-\tan^{-1}\alpha}^{0} \mathrm{d}y\, y^3\, \sec^6 y, \tag{4.20}$$

which can be evaluated using the result [58]

$$\begin{aligned}
\int \mathrm{d}y\, y^3\, \sec^6 y &= -\frac{y^2\left(3\,\cos y - 4\,y\,\sin y\right)}{20\,\cos^5 y} - \frac{2\,y^2\left(3\,\cos y - 2\,y\,\sin y\right)}{15\,\cos^3 y}\\
&\quad + \left(y + \frac{8\,y^3}{15}\right)\tan y + \ln|\cos y| + \frac{y\,\sin y}{10\,\cos^3 y} - \frac{1}{20\,\cos^2 y}\\
&\quad + \frac{8}{5}\sum_{n=1}^{\infty}\frac{(-1)^n\,2^{2\,n}\left(2^{2\,n}-1\right)\,y^{2\,n+2}}{(2\,n+2)\,(2\,n)!}\,B_{2n}. \qquad (4.21)
\end{aligned}$$

The infinite series in the above expression is convergent, and it can be expressed as [59]

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^n\, 2^{2\,n}\, \left(2^{2\,n}-1\right)\, y^{2\,n+2}}{(2\,n+2)\,(2\,n)!}\, B_{2n} = y^2 \biggl\{ & \ln\left[\Gamma\left(1+\frac{y}{\pi}\right)\right] + \ln\left[\Gamma\left(1-\frac{y}{\pi}\right)\right] \\
& - \ln\left[\Gamma\left(1-\frac{2y}{\pi}\right)\right] - \ln\left[\Gamma\left(1+\frac{2y}{\pi}\right)\right]\biggr\} + \frac{3}{8}\,\zeta(3) \\
& + \pi^2 \left[\zeta'\left(-2, 1+\frac{y}{\pi}\right) + \zeta'\left(-2, 1-\frac{y}{\pi}\right)\right. \\
& \left. - \frac{1}{4}\,\zeta'\left(-2, 1-\frac{2y}{\pi}\right) - \frac{1}{4}\,\zeta'\left(-2, 1+\frac{2y}{\pi}\right)\right] \\
& + \pi\, y \left[\zeta'\left(-1, 1+\frac{2y}{\pi}\right) - \zeta'\left(-1, 1-\frac{2y}{\pi}\right)\right. \\
& \left. + 2\,\zeta'\left(-1, 1-\frac{y}{\pi}\right) - 2\,\zeta'\left(-1, 1+\frac{y}{\pi}\right)\right], \qquad (4.22)
\end{aligned}
$$

where, as before, $\zeta(s)$ is the Riemann zeta function, $\zeta'(s,a)$ denotes the derivative of the Hurwitz zeta function $\zeta(s,a)$ with respect to the first argument and $\Gamma(n)$ is the Gamma function.

Let us now consider the quantity $\bar{\mathcal{G}}_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ in the third domain, *i.e.* from the bounce at $\eta = 0$ to $\eta = \beta\,\eta_0$. In this domain, the modes h_k and the scale factor have the same form as in the second domain. Therefore, it should be clear that, the quantity $\bar{\mathcal{G}}^3_{\gamma\gamma\gamma}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ too will be given by the expression (4.5), but with the integrals $J_n(\alpha)$ being replaced by $-J_n(-\beta)$.

5 Results

We can now make use of the behavior of the mode h_k at $\eta_{\rm e} = \beta\,\eta_0$ and substitute the results we have obtained above in the expressions (3.6) and (3.8) to arrive at the tensor bi-spectrum and the corresponding non-Gaussianity parameter $h_{_{\rm NL}}$ for an arbitrary triangular configuration of the wavenumbers involved. The resulting expressions prove to be rather long and, for this reason, we shall illustrate the various results graphically for a set of suitable values of the parameters. Let us first compare the contributions from the three domains. Restricting ourselves to the equilateral limit, in Fig. 3, we have plotted the contributions to $h_{_{\rm NL}}$ from the three domains that we have considered. It is evident from the figure that the contribution due to the third domain to the parameter $h_{_{\rm NL}}$ turns out to be the maximum. We find that the third domain contributes the maximum in the squeezed limit as well.

It is now a matter of adding the contributions from the three domains to arrive at the complete tensor bi-spectrum and the non-Gaussianity parameter $h_{_{\rm NL}}$. In Fig. 4, we have plotted the behavior of the non-Gaussianity parameter $h_{_{\rm NL}}$ in the equilateral and the squeezed limits. Three points concerning the figure require emphasis. To begin with, we should mention that the non-Gaussianity parameter $h_{_{\rm NL}}$ behaves as k^2 in both the equilateral and the squeezed limits, with virtually the same amplitude. Secondly, the value of the parameter $h_{_{\rm NL}}$ is very small when compared to the values that occur in, say, de Sitter inflation wherein $3/8 \lesssim h_{_{\rm NL}} \lesssim 1/2$ (in this context, see Ref. [34]). Thirdly, since the tensor power spectrum

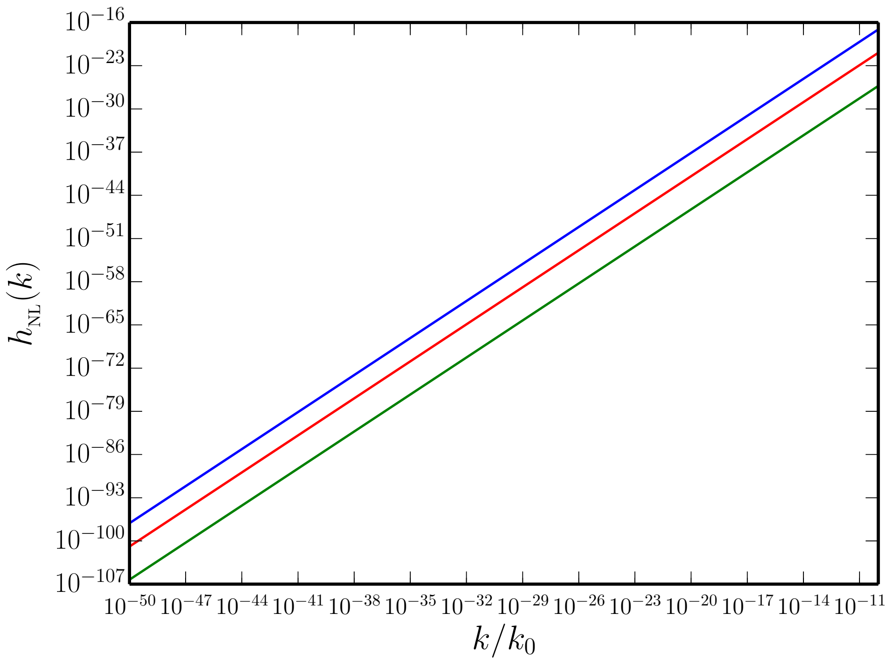

Figure 3. The contributions to the non-Gaussianity parameter $h_{\rm NL}$ in the equilateral limit from the first (in green), the second (in red) and the third (in blue) domains have been plotted as a function of k/k_0 for a wide range of wavenumbers such that $k \ll k_0/\alpha$. We have worked with the same set of values as in the previous figure. Clearly, the third domain gives rise to the maximum contribution to the non-Gaussianity parameter $h_{\rm NL}$.

is strictly scale invariant for wavenumbers such that $k \ll k_0/\alpha$, the amplitude of the non-Gaussianity parameter $h_{\rm NL}$ in the squeezed limit over such domain should be equal to $3/8$, if the consistency relation holds true (see Appendix, also see Ref. [45]). Whereas, we find that $h_{\rm NL}$ is considerably smaller than $3/8$ in the squeezed limit, which unambiguously implies that the consistency condition is violated. Evidently, this behavior can be attributed to the fact that the amplitude of the tensor mode does not freeze to a constant value at late times.

At this stage, we need to discuss the dependence of the tensor power and bi-spectra on the parameters α and β that we have introduced. We find that the tensor power spectrum and the bi-spectrum do not significantly depend on α over a wide range of values, say, $10^5 \lesssim \alpha \lesssim 10^{15}$. We had mentioned earlier that the tensor power spectrum has a rather weak dependence on β. Whereas, we find that the tensor bi-spectrum grows roughly as $\beta^{3/2}$. However, β cannot be allowed to be too large for two reasons. One may a priori expect that the analytical approximation (2.16) will remain valid until the time $-\eta_k = \sqrt{2}/k$ after the bounce. We had pointed out that the evolution of the mode h_k is asymmetric in η. Actually, it can be shown that (using numerical analysis) the analytical approximation (2.16) breaks down much before $-\eta_k$. For this reason, we have chosen β to be smaller than α. Moreover, a transition to the radiation dominated phase is expected to take place sometime after the bounce. It seems reasonable to expect that such a transition will occur when the scale factor is $a \simeq 10^4\, a_0$, which corresponds to $\beta = 10^2$. We find that our main conclusions, *viz.* that the value of $h_{\rm NL}$ is small over cosmological scales and that the consistency relation is violated in the squeezed limit, continue to remain valid even if we increase β by, say, a couple of orders of magnitude.

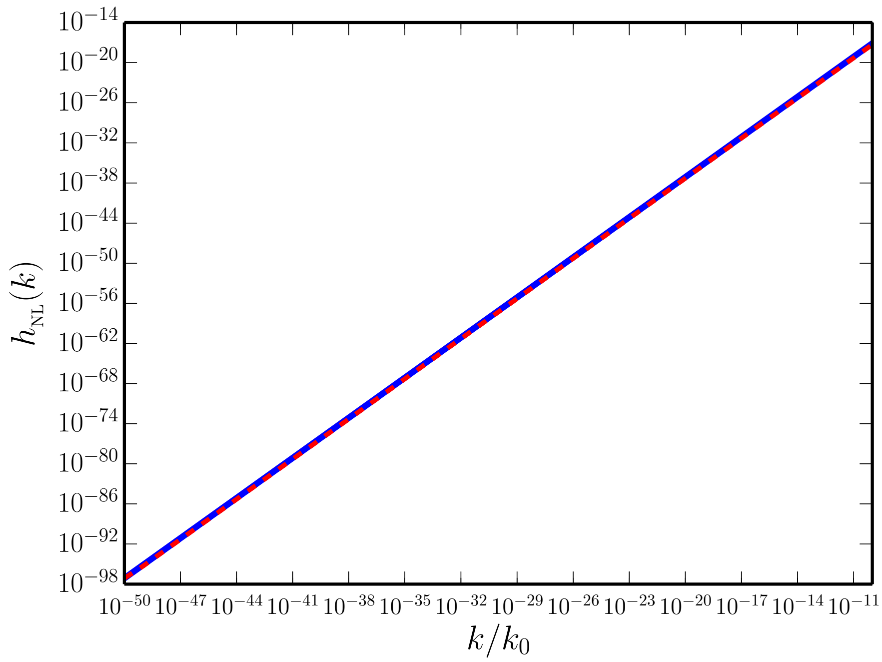

Figure 4. The behavior of the non-Gaussianity parameter $h_{_{\rm NL}}$ in the equilateral (in blue) and the squeezed (in red) limits have been plotted as a function of k/k_0 for a wide range of wavenumbers such that $k \ll k_0/\alpha$. We have worked with the same set of values as in the earlier two figures. Clearly, the resulting $h_{_{\rm NL}}$ is considerably small when compared to the values that arise in de Sitter inflation wherein $3/8 \lesssim h_{_{\rm NL}} \lesssim 1/2$. Moreover, we find that $h_{_{\rm NL}}$ behaves as k^2 in the equilateral and the squeezed limits, with similar amplitudes. The fact that $h_{_{\rm NL}}$ is much smaller than 3/8 in the squeezed limit implies that the consistency condition is violated.

6 Discussion

In this paper, we have analytically calculated the tensor bi-spectrum in a matter bounce using the Maldacena formalism. While the matter bounce leads to a scale invariant tensor power spectrum for scales of cosmological interest as de Sitter inflation does, we have shown that the non-Gaussianity parameter $h_{_{\rm NL}}$ that characterizes the amplitude of the tensor bi-spectrum is much smaller than the corresponding values in de Sitter inflation. We have also shown that, due to the growth in amplitude of the tensor modes as one approaches the bounce, the consistency condition is not satisfied by the tensor bi-spectrum in the squeezed limit. Recall that, in the absence of detailed modelling of the bounce, we had assumed that $k_0 \simeq M_{_{\rm Pl}}$. We should however clarify that, since $k \ll k_0$ for cosmological scales, our essential conclusions, *viz.* that $h_{_{\rm NL}}$ is small and that the consistency condition is violated over such scales, will remain unaffected even if we choose k_0 to be a few orders of magnitude below the Planck scale. It will clearly be worthwhile to investigate these issues using a numerical approach in a wider class of bouncing models.

In the bouncing scenario that we have considered, at very early times, *i.e.* during the first domain of our interest, the contribution to the non-Gaussianity parameter $h_{_{\rm NL}}$ can be said to be small because the amplitude of the tensor perturbations themselves are small. In the second domain, although the scale factor decreases gradually to reach its minimum at the bounce, the non-Gaussianities become larger as the perturbations grow. In the third

domain, *i.e.* after the bounce, the scale factor increases steadily. Also, the amplitude of the perturbations do not freeze but grow slowly. Due to these reasons the contribution to the non-Gaussianity parameter is the largest from this regime. However, essentially due to the form of the scale factor, one finds that the parameter $h_{\rm NL}$ has an overall k^2 dependence. Since the scales of cosmological interest are about 50 to 60 orders below the Planck scale, the non-Gaussianity parameter $h_{\rm NL}$ proves to be very small over such scales.

We believe that the results we have obtained have tremendous implications for the other three-point functions and, importantly, the scalar bi-spectrum. It seems clear that, due to the growth during the contracting phase near the bounce, the consistency conditions governing the other three-point functions will be violated as well [24]. This possibly can act as a powerful discriminator between the inflationary and bouncing scenarios. Within inflation, one requires peculiar situations to violate the consistency conditions [60, 61]. In contrast, in a bouncing scenario, the consistency relations seem to be violated rather naturally. Notably, situations involving violations of the consistency conditions have been considered as possible sources of spherical asymmetry in the early universe [28–30]. These aspects seem worth exploring in greater detail.

Acknowledgements

The authors wish to thank Robert Brandenberger, Dhiraj Hazra, Jérôme Martin and Patrick Peter for discussions and comments on the manuscript. VS would like to acknowledge Department of Science and Technology, India, for supporting this work through the project PHY1213274DSTXDAWO. LS wishes to thank the Indian Institute of Technology Madras, Chennai, India, for support through the New Faculty Seed Grant.

Appendix: The squeezed limit and the consistency condition

An important property of the three-point functions is their behavior in the so-called squeezed limit [27, 35–43]. As we have discussed before, the squeezed limit corresponds to the situation wherein one of the three wavenumbers is much smaller than the other two. In such a limit, under certain conditions, it is known that all the three-point functions involving the scalars and tensors generated during inflation can be expressed entirely in terms of the two-point functions [28–30, 44, 45]. In the context of inflation, these consistency conditions arise essentially because of the fact that the amplitude of the long wavelength scalar and tensor modes freeze on super-Hubble scales. In this appendix, we shall outline the proof of the consistency condition satisfied by the tensor bi-spectrum during inflation.

Since the amplitude of a long wavelength mode freezes on super-Hubble scales during inflation, such modes can be treated as a background as far as the smaller wavelength modes are concerned. Let us denote the constant amplitude of the long wavelength tensor mode as $\gamma_{ij}^{\rm B}$. In the presence of such a long wavelength mode, the background FLRW metric can be written as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\, [\mathrm{e}^{\gamma^{\rm B}}]_{ij}\, \mathrm{d}\boldsymbol{x}^i\, \mathrm{d}\boldsymbol{x}^j, \tag{.1}$$

i.e. the spatial coordinates are modified according to a spatial transformation of the form $\boldsymbol{x}' = \Lambda\, \boldsymbol{x}$, where the components of the matrix Λ are given by $\Lambda_{ij} = [\mathrm{e}^{\gamma^{\rm B}/2}]_{ij}$. Under such a spatial transformation, the small wavelength tensor perturbation transforms as [45]

$$\gamma_{ij}^{\boldsymbol{k}} \to \det\,(\Lambda^{-1})\, \gamma_{ij}^{\Lambda^{-1}\boldsymbol{k}}, \tag{.2}$$

where $\det(\Lambda^{-1}) = 1$. Under these conditions, we also obtain that

$$|\Lambda^{-1}\,\boldsymbol{k}| = [1 - \gamma^{\rm B}_{ij}\,k_i\,k_j/(2\,k^2)]\,k, \tag{.3}$$

where k_i is the component of the wavevector $\boldsymbol{k}$ along the i-spatial direction and we have restricted ourselves to the leading order in $\gamma^{\rm B}_{ij}$. Moreover, one can show that

$$\delta^{(3)}(\Lambda^{-1}\,\boldsymbol{k}_1 + \Lambda^{-1}\,\boldsymbol{k}_2) = \det(\Lambda)\,\delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2) = \delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2), \tag{.4}$$

since $\det(\Lambda) = 1$. Upon using the above results, we find that the tensor two-point function in the presence of a long wavelength mode can be written as

$$\begin{aligned}\langle \hat{\gamma}^{\boldsymbol{k}_1}_{m_1n_1}\,\hat{\gamma}^{\boldsymbol{k}_2}_{m_2n_2}\rangle_k = &\frac{(2\,\pi)^2}{2\,k_1^3}\,\frac{\Pi^{\boldsymbol{k}_1}_{m_1n_1,m_2n_2}}{4}\,\mathcal{P}_{_{\rm T}}(k_1)\,\delta^{(3)}(\boldsymbol{k}_1 + \boldsymbol{k}_2)\\ &\times\left[1 - \left(\frac{n_{_{\rm T}} - 3}{2}\right)\,\gamma^{\rm B}_{ij}\,\hat{n}_{1i}\,\hat{n}_{1j}\right].\end{aligned} \tag{.5}$$

where $\hat{n}_{1i} = k_{1i}/k_1$ and the long wavelength mode is denoted by the wavenumber k, while $n_{_{\rm T}}$ represents the tensor spectral index. The corresponding expression for the tensor bi-spectrum can be obtained from the above result to be

$$\begin{aligned}\langle\,\hat{\gamma}^{\boldsymbol{k}_1}_{m_1n_1}\,\hat{\gamma}^{\boldsymbol{k}_2}_{m_2n_2}\,\hat{\gamma}^{\boldsymbol{k}_3}_{m_3n_3}\,\rangle_{k_3} &\equiv \langle\,\langle\,\hat{\gamma}^{\boldsymbol{k}_1}_{m_1n_1}\,\hat{\gamma}^{\boldsymbol{k}_2}_{m_2n_2}\,\rangle_{k_3}\,\hat{\gamma}^{\boldsymbol{k}_3}_{m_3n_3}\,\rangle\\ &= -\,\frac{(2\,\pi)^{5/2}}{4\,k_1^3\,k_3^3}\,\left(\frac{n_{_{\rm T}} - 3}{32}\right)\,\mathcal{P}_{_{\rm T}}(k_1)\,\mathcal{P}_{_{\rm T}}(k_3)\\ &\quad\times\Pi^{\boldsymbol{k}_1}_{m_1n_1,m_2n_2}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,ij}\,\hat{n}_{1i}\,\hat{n}_{1j}\,\delta^3(\boldsymbol{k}_1 + \boldsymbol{k}_2),\end{aligned} \tag{.6}$$

where $\boldsymbol{k_3}$ has been considered to be the squeezed mode. The above relation wherein the tensor bi-spectrum has been expressed completely in terms of the power spectrum is known as the consistency condition [44, 45]. Upon substituting this expression in the definition for the tensor non-Gaussianity parameter $h_{_{\rm NL}}$ [cf. Eq. (3.5)], we find that we can express the consistency relation in the squeezed limit as follows:

$$\begin{aligned}\lim_{k_3\to 0}\,h_{_{\rm NL}}(\boldsymbol{k}, -\boldsymbol{k}, \boldsymbol{k}_3) = &\left[\frac{n_{_{\rm T}}(k) - 3}{2}\right]\,\Big(2\,\Pi^{\boldsymbol{k}}_{m_1n_1,m_2n_2}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,\bar{m}\bar{n}} + \Pi^{\boldsymbol{k}}_{m_1n_1,\bar{m}\bar{n}}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,m_2n_2}\\ &+ \Pi^{\boldsymbol{k}}_{\bar{m}\bar{n},m_2n_2}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,m_1n_1}\Big)^{-1}\,\Pi^{\boldsymbol{k}}_{m_1n_1,m_2n_2}\,\Pi^{\boldsymbol{k}_3}_{m_3n_3,ij}\,\hat{n}_i\,\hat{n}_j.\end{aligned} \tag{.7}$$

Actually, an overall minus sign occurs in the expression for $h_{_{\rm NL}}$ in the squeezed limit due to the polarization tensors [45], Therefore, if we ignore the polarization tensors, in the domain where the tensor power spectrum is strictly scale invariant (*i.e.* when $n_{_{\rm T}} = 0$), the value of $h_{_{\rm NL}}$ in the squeezed limit reduces to $3/8$, if the consistency relation is satisfied.

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